

ON SOME INEQUALITIES FOR DIFFERENT KINDS OF CONVEXITY

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ABSTRACT. In this paper, we obtained some inequalities for φ_s -convex function, φ -Godunova-Levin function, φ - P -function and \log - φ -convex function. Finally, we defined the class of φ -*quasi*-convex functions and we examined some properties of this class.

1. INTRODUCTION

Let us consider a function $\varphi : [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$.

In [1] and [2], M.Z. Sarikaya defined the following classes:

Definition 1. Let I be an interval in \mathbb{R} and $h : (0, 1) \rightarrow (0, \infty)$ be a given function. We say that a function $f : I \rightarrow [0, \infty)$ is φ_h -convex if

$$(1.1) \quad f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y))$$

for all $x, y \in I$ and $t \in (0, 1)$. If inequality (1.1) is reversed, then f is said to be φ_h -concave. In particular if f satisfies (1.1) with $h(t) = t$, $h(t) = t^s$ ($s \in (0, 1)$), $h(t) = \frac{1}{t}$ and $h(t) = 1$, then f is said to be φ -convex, φ_s -convex, φ -Godunova-Levin function and φ - P -function, respectively.

Definition 2. Let us consider a $\varphi : [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$ and I stands for a convex subset of \mathbb{R} . We say that a function $f : I \rightarrow \mathbb{R}^+$ is a \log - φ -convex if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1-t}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In this paper, we examined the character of the function $f \circ \varphi$ according to character of f and φ functions and we obtained inequalities for \log - φ -convex function, φ_s -convex function, φ -Godunova-Levin function and φ - P -function. Finally we defined φ -*quasi*-convex functions and we gave some properties of this class.

2. MAIN RESULTS

Theorem 1. Let f be φ_s -convex function. Then i) if φ is linear, then $f \circ \varphi$ is s -convex in the second sense and ii) if f is increasing and φ is convex, then $f \circ \varphi$ is s -convex in the second sense.

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Proof. i) From φ_s -convexity of f and linearity of φ , we have

$$\begin{aligned} f \circ \varphi [\lambda x + (1 - \lambda)y] &= f [\varphi (\lambda x + (1 - \lambda)y)] \\ &= f [\lambda \varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq \lambda^s f(\varphi(x)) + (1 - \lambda)^s f(\varphi(y)) \end{aligned}$$

which completes the proof for first case.

ii) From convexity of φ , we have

$$\varphi [\lambda x + (1 - \lambda)y] \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f \circ \varphi [\lambda x + (1 - \lambda)y] &\leq f [\lambda \varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq \lambda^s f(\varphi(x)) + (1 - \lambda)^s f(\varphi(y)). \end{aligned}$$

This completes the proof for this case. \square

Theorem 2. Let f be φ_s -convex and let $\sum_{i=1}^n t_i = T_n = 1$, $t_i \in (0, 1)$, $i = 1, 2, \dots, n$, $s \in (0, 1)$, then

$$f \left(\sum_{i=1}^n t_i \varphi(x_i) \right) \leq \sum_{i=1}^n t_i^s f(\varphi(x_i)).$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f \left(\sum_{i=1}^n t_i \varphi(x_i) \right) &= f \left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i) + t_n \varphi(x_n) \right) \\ &\leq (T_{n-1})^s f \left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i) \right) + t_n^s f(\varphi(x_n)) \\ &= (T_{n-1})^s f \left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i) + \frac{t_{n-1}}{T_{n-1}} \varphi(x_{n-1}) \right) + t_n^s f(\varphi(x_n)) \\ &\leq (T_{n-2})^s f \left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i) \right) + t_{n-1}^s f(\varphi(x_{n-1})) + t_n^s f(\varphi(x_n)) \\ &\quad \vdots \\ &\leq \sum_{i=1}^n t_i^s f(\varphi(x_i)). \end{aligned}$$

This completes the proof. \square

Theorem 3. Let f be φ -Godunova-Levin function. Then i) if φ is linear, then $f \circ \varphi$ belongs to $Q(I)$ and ii) if f is increasing and φ is convex, then $f \circ \varphi \in Q(I)$.

Proof. i) Since f is φ -Godunova-Levin function and from linearity of φ , we have

$$\begin{aligned} f \circ \varphi [\lambda x + (1 - \lambda)y] &= f [\varphi (\lambda x + (1 - \lambda)y)] \\ &= f [\lambda \varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq \frac{f \circ \varphi(x)}{\lambda} + \frac{f \circ \varphi(y)}{1 - \lambda} \end{aligned}$$

which completes the proof.

ii) From convexity of φ , we have

$$\varphi[\lambda x + (1 - \lambda)y] \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f \circ \varphi[\lambda x + (1 - \lambda)y] &\leq f[\lambda\varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq \frac{f \circ \varphi(x)}{\lambda} + \frac{f \circ \varphi(y)}{1 - \lambda}. \end{aligned}$$

This completes the proof. \square

Theorem 4. Let f be φ -Godunova-Levin function and let $\sum_{i=1}^n t_i = T_n = 1$, $t_i \in (0, 1)$, $i = 1, 2, \dots, n$, then

$$f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) \leq \sum_{i=1}^n \frac{f(\varphi(x_i))}{t_i}.$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) &= f\left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i) + t_n \varphi(x_n)\right) \\ &\leq \frac{f\left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i)\right)}{T_{n-1}} + \frac{f(\varphi(x_n))}{t_n} \\ &= \frac{1}{T_{n-1}} f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i) + \frac{t_{n-1}}{T_{n-1}} \varphi(x_{n-1})\right) + \frac{f(\varphi(x_n))}{t_n} \\ &\leq \frac{f\left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i)\right)}{T_{n-2}} + \frac{f(\varphi(x_{n-1}))}{t_{n-1}} + \frac{f(\varphi(x_n))}{t_n} \\ &\vdots \\ &\leq \sum_{i=1}^n \frac{f(\varphi(x_i))}{t_i}. \end{aligned}$$

This completes the proof. \square

Theorem 5. Let f be φ - P -convex function. Then i) if φ is linear, then $f \circ \varphi$ belongs to $P(I)$ and ii) if f is increasing and φ is convex, then $f \circ \varphi \in P(I)$.

Proof. i) From φ - P -convexity of f and linearity of φ , we have

$$\begin{aligned} f \circ \varphi[\lambda x + (1 - \lambda)y] &= f[\varphi(\lambda x + (1 - \lambda)y)] \\ &= f[\lambda\varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq f(\varphi(x)) + f(\varphi(y)), \end{aligned}$$

which completes the proof.

ii) From convexity of φ , we have

$$\varphi[\lambda x + (1 - \lambda)y] \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f \circ \varphi[\lambda x + (1 - \lambda)y] &\leq f[\lambda\varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq f(\varphi(x)) + f(\varphi(y)). \end{aligned}$$

This completes the proof. \square

Theorem 6. Let f be $\varphi - P$ -convex and let $\sum_{i=1}^n t_i = T_n = 1$, $t_i \in (0, 1)$, $i = 1, 2, \dots, n$, then

$$f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) \leq \sum_{i=1}^n f(\varphi(x_i)).$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) &= f\left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i) + t_n \varphi(x_n)\right) \\ &\leq f\left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i)\right) + f(\varphi(x_n)) \\ &= f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i) + \frac{t_{n-1}}{T_{n-1}} \varphi(x_{n-1})\right) + f(\varphi(x_n)) \\ &\leq f\left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i)\right) + f \circ \varphi(x_{n-1}) + f(\varphi(x_n)) \\ &\quad \vdots \\ &\leq \sum_{i=1}^n f(\varphi(x_i)). \end{aligned}$$

This completes the proof. \square

Theorem 7. Let f be $\log - \varphi$ -convex function. Then i) if φ is linear, then $f \circ \varphi$ is $\log -$ convex and ii) if f is increasing and φ is convex, then $f \circ \varphi$ is $\log -$ convex function.

Proof. i) From $\log - \varphi$ -convexity of f and linearity of φ , we have

$$\begin{aligned} f \circ \varphi [\lambda x + (1 - \lambda)y] &= f [\varphi (\lambda x + (1 - \lambda)y)] \\ &= f [\lambda \varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq [f(\varphi(x))]^\lambda [f(\varphi(y))]^{1-\lambda} \end{aligned}$$

which completes the proof for first case.

ii) From convexity of φ , we have

$$\varphi [\lambda x + (1 - \lambda)y] \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f \circ \varphi [\lambda x + (1 - \lambda)y] &\leq f [\lambda \varphi(x) + (1 - \lambda)\varphi(y)] \\ &\leq [f(\varphi(x))]^\lambda [f(\varphi(y))]^{1-\lambda}. \end{aligned}$$

This completes the proof for this case. \square

Theorem 8. Let f be $\log - \varphi$ -convex function. For $a, b \in I$ with $a < b$ and $\lambda \in [0, 1]$, one has the inequality

$$\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x)) dx \leq G(f(\varphi(a)), f(\varphi(b))).$$

where $G(\cdot, \cdot)$ is the geometric mean.

Proof. Since f is log φ -convex function, we have that

$$\begin{aligned} f(\lambda\varphi(a) + (1-\lambda)\varphi(b)) &\leq [f(\varphi(a))]^\lambda [f(\varphi(b))]^{1-\lambda} \\ f((1-\lambda)\varphi(a) + \lambda\varphi(b)) &\leq [f(\varphi(a))]^{1-\lambda} [f(\varphi(b))]^\lambda \end{aligned}$$

for all $\lambda \in [0, 1]$.

If we multiply the above inequalities and take square roots, we obtain

$$G(f(\lambda\varphi(a) + (1-\lambda)\varphi(b)), f((1-\lambda)\varphi(a) + \lambda\varphi(b))) \leq G(f(\varphi(a)), f(\varphi(b))).$$

Integrating this inequality over λ on $[0, 1]$, and changing the variable $x = \lambda\varphi(a) + (1-\lambda)\varphi(b)$, we have

$$\begin{aligned} &\int_0^1 G(f(\lambda\varphi(a) + (1-\lambda)\varphi(b)), f((1-\lambda)\varphi(a) + \lambda\varphi(b))) d\lambda \\ &\leq G(f(\varphi(a)), f(\varphi(b))), \\ &\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x)) dx \leq G(f(\varphi(a)), f(\varphi(b))) \end{aligned}$$

which completes the proof. \square

Definition 3. Let us consider a $\varphi : [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$ and I stands for a convex subset of \mathbb{R} . We say that a function $f : I \rightarrow \mathbb{R}^+$ is a φ -quasi-convex if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq \max\{f(\varphi(x)), f(\varphi(y))\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

Theorem 9. Let f be φ -quasi-convex function. Then i) if φ is linear, then $f \circ \varphi$ is quasi-convex and ii) if f is increasing and φ is convex, then $f \circ \varphi$ is quasi-convex function.

Proof. i) From φ -quasi-convexity of f and linearity of φ , we have

$$\begin{aligned} f \circ \varphi[\lambda x + (1-\lambda)y] &= f[\varphi(\lambda x + (1-\lambda)y)] \\ &= f[\lambda\varphi(x) + (1-\lambda)\varphi(y)] \\ &\leq \max\{f(\varphi(x)), f(\varphi(y))\}, \end{aligned}$$

which completes the proof for first case.

ii) From convexity of φ , we have

$$\varphi[\lambda x + (1-\lambda)y] \leq \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

Since f is increasing we can write

$$\begin{aligned} f \circ \varphi[\lambda x + (1-\lambda)y] &\leq f[\lambda\varphi(x) + (1-\lambda)\varphi(y)] \\ &\leq \max\{f(\varphi(x)), f(\varphi(y))\}. \end{aligned}$$

This completes the proof for this case. \square

Theorem 10. Let f be φ -quasi-convex function. For $x, y \in [a, b]$, $x < y$ and $\lambda \in [0, 1]$, one has the inequality

$$(2.1) \quad \frac{1}{\varphi(y) - \varphi(x)} \int_{\varphi(x)}^{\varphi(y)} f(u) du \leq \max\{f(\varphi(x)), f(\varphi(y))\}.$$

Proof. Since f is φ -quasi-convex function, we can write

$$f[\lambda\varphi(x) + (1-\lambda)\varphi(y)] \leq \max\{f(\varphi(x)), f(\varphi(y))\}$$

and

$$f[(1-\lambda)\varphi(x) + \lambda\varphi(y)] \leq \max\{f(\varphi(x)), f(\varphi(y))\}.$$

If we add the above inequalities and integrate on $[0, 1]$, we have

$$\begin{aligned} & \frac{1}{2} \int_0^1 [f[\lambda\varphi(x) + (1-\lambda)\varphi(y)] + f[(1-\lambda)\varphi(x) + \lambda\varphi(y)]] d\lambda \\ & \leq \max\{f(\varphi(x)), f(\varphi(y))\} \end{aligned}$$

which is equal to inequality in (2.1). \square

Theorem 11. Let f be φ -quasi-convex and let $\sum_{i=1}^n t_i = T_n = 1$, $t_i \in (0, 1)$, $i = 1, 2, \dots, n$, then

$$f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) \leq \max_{1 \leq i \leq n} f(\varphi(x_i)).$$

Proof. From the above assumptions, we can write

$$\begin{aligned} f\left(\sum_{i=1}^n t_i \varphi(x_i)\right) &= f\left(T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i) + t_n \varphi(x_n)\right) \\ &\leq \max\left\{f\left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \varphi(x_i)\right), f(\varphi(x_n))\right\} \\ &= \max\left\{f\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i) + \frac{t_{n-1}}{T_{n-1}} \varphi(x_{n-1})\right), f(\varphi(x_n))\right\} \\ &\leq \max\left\{f\left(\sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \varphi(x_i)\right), f(\varphi(x_{n-1})), f(\varphi(x_n))\right\} \\ &\vdots \\ &\leq \max\{f(\varphi(x_1)), \dots, f(\varphi(x_{n-1})), f(\varphi(x_n))\} \\ &= \max_{1 \leq i \leq n} f(\varphi(x_i)). \end{aligned}$$

This completes the proof. \square

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